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Changes of signs in conditionally convergent series on a small set

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ABSTRACT

We consider ideals I of subsets of the set of natural numbers $\mathbb{N}$ such that for every conditionally convergent series of real numbers $\sum_{n\in\mathbb{N}} a_n$ and $s \in \overline{\mathbb{R}}$, then there is a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n a_n = s$ and $N(\delta) := \{n \in \mathbb{N} : \delta_n = -1\} \in I$. We give some properties of such ideals and characterize them in terms of extendability to a summable ideal.

1. Introduction

A celebrated theorem of Riemann affirms that if the series of real numbers $\sum_{n\in\mathbb{N}} a_n$ is conditionally convergent and if $s \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, then there is a permutation π of the set $\mathbb{N}$ of all natural numbers such that $\sum_{n\in\mathbb{N}} a_{\pi(n)} = s$ (for example, [1, Theorem 23.7]).

Filipów and Szuca [2] have studied whether it is possible to take the permutation π, in the above theorem of Riemann, such that it changes only a "small set" of terms of the series, where the notion of smallness is induced by an ideal of subsets of the set $\mathbb{N}$ (for more information about set ideals see [2–4]). It is said that the ideal I has the (R) *property* if for any conditionally convergent series of real numbers $\sum_{n\in\mathbb{N}} a_n$ and $s \in \overline{\mathbb{R}}$, then there is a permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} a_\pi(n) = s$ and $\{n \in \mathbb{N} : \pi(n) \neq n\} \in I$.

We consider a similar situation. It is known that if the series of real numbers $\sum_{n\in\mathbb{N}} a_n$ is conditionally convergent and if $s \in \overline{\mathbb{R}}$, then there exists a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n a_n = s$ (Corollary 2.7). We say that the ideal I has the (S) *property* if it is possible to take δ such that $N(\delta) := \{n \in \mathbb{N} : \delta_n = -1\}$ is in I (see Definition 2.5).

In this note we prove that the (R) and (S) properties are equivalent (Theorem 3.3). That is, for an ideal I of subsets of $\mathbb{N}$, given every conditionally convergent series $\sum_{n\in\mathbb{N}} a_n$ and any $s \in \overline{\mathbb{R}}$, there is a permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} a_{\pi(n)} = s$ and $\{n \in \mathbb{N} : \pi(n) \neq n\} \in I$ if and only if there exists a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n a_n = s$ and $N(\delta) := \{n \in \mathbb{N} : \delta_n = -1\} \in I$.

The ideal $\mathcal{I}_d$ of all the sets of asymptotic density zero has the (R) property [4]; hence it has the (S) property (Corollary 3.4). Recall that a subset A of $\mathbb{N}$ belongs to the ideal $\mathcal{I}_d$ if

$$\limsup_{n\to\infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n} = 0,$$

where $|B|$ denotes the cardinal of B [2, Example 2.3].

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2. Preliminaries

We begin with the definition of an ideal:

Definition 2.1. A family I of subsets of $\mathbb{N}$ is an *ideal* on $\mathbb{N}$ if:

(1) If $A \in I$ and $B \subset A$, then $B \in I$.
(2) If $A, B \in I$, then $A \cup B \in I$.

The ideal I is *proper* if $\mathbb{N}$ does not belong to I and it is *admissible* if any finite subset of $\mathbb{N}$ is in I.

In this paper all the ideals are proper and admissible. We need the notion of a summable ideal which is based on the following example.

Example 2.2. Let $b = (b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{n\in\mathbb{N}} b_n = \infty$. The family

$$I_b = \left\{ A \subset \mathbb{N} : \sum_{n\in A} b_n < \infty \right\}$$

is an ideal on $\mathbb{N}$.

Now some types of ideal are defined:

Definition 2.3 ([2]). Let I be an ideal proper and admissible on $\mathbb{N}$. Then

(1) I has the (R) *property* if given the conditionally convergent series of real numbers $\sum_{n\in\mathbb{N}} a_n$ and $s \in \overline{\mathbb{R}}$, then there is a permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} a_{\pi(n)} = s$ and $\{n \in \mathbb{N} : \pi(n) \neq n\} \in I$.
(2) I has the (W) *property* if given the conditionally convergent series of real numbers $\sum_{n\in\mathbb{N}} a_n$, then there exists $A \in I$ such that $\sum_{n\in A} a_n$ is also conditionally convergent.
(3) I is *summable* if $I = I_b$ for some sequence $b = (b_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{n\in\mathbb{N}} b_n = \infty$ (see Example 2.2).
(4) I is *dense* if every $A \subset \mathbb{N}$, with $A \notin I$, contains an infinite subset that belongs to I.

In [4, Question 1], Wilczyński asked for a characterization of ideals which have the (W) property. Recently, Filipów and Szuca proved that the (R) and (W) properties are equivalent, and they are also equivalent to the property that the ideal I cannot be extended to a summable ideal.

Theorem 2.4 ([2, Theorem 3.3]). *Let I be an admissible and proper ideal on $\mathbb{N}$. The following are equivalent:*

(1) *I has the (R) property.*
(2) *I cannot be extended to a summable ideal.*
(3) *I has the (W) property.*

We introduce the following concept, which is central in this paper:

Definition 2.5. Let I be an ideal proper and admissible on $\mathbb{N}$. We say that I has the (S) *property* if given the conditionally convergent series of real numbers $\sum_{n\in\mathbb{N}} a_n$ and $s \in \overline{\mathbb{R}}$, then there is a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n a_n = s$ and $N(\delta) := \{n \in \mathbb{N} : \delta_n = -1\} \in I$.

The following result concerning series of real numbers is basic for our purpose. We give a proof since we have not found one in any text.

Theorem 2.6. *Let $b = (b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers converging to zero such that $\sum_{n\in\mathbb{N}} b_n = \infty$ and suppose that $s \in \overline{\mathbb{R}}$. Then there exists a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n b_n = s$.*

Proof. Assume that $0 \leq s < \infty$. The other cases are proved in a similar way. For any natural number n we define

$$s_n := \delta_1 b_1 + \cdots + \delta_n b_n,$$

the n-partial sum of the series $\sum_{n\in\mathbb{N}} \delta_n b_n$, where δ_n is defined further on. Let n_1 be the smallest natural number such that

$$s_{n_1} = \sum_{1\leq n\leq n_1} b_n = \sum_{1\leq n\leq n_1} \delta_n b_n > s,$$

where $\delta_n = 1$ for $1 \leq n \leq n_1$. Let n_2 be the smallest natural number such that $n_2 > n_1$ and

$$s_{n_2} = s_{n_1} - \sum_{n_1+1\leq n\leq n_2} b_n = \sum_{1\leq n\leq n_2} \delta_n b_n \leq s,$$

that is $\delta_n = -1$ for $n_1 + 1 \le n \le n_2$. For every natural number n with $n_1 \le n < n_2$ we have that

$$|s_n - s| = s_n - s \le s_{n_1} - s < b_{n_1}.$$

Let n_3 be the smallest natural number such that $n_3 > n_2$ and

$$s_{n_3} = s_{n_2} + \sum_{n_2+1 \le n \le n_3} a_n = \sum_{1 \le n \le n_3} \delta_n b_n > s,$$

where $\delta_n = 1$ for $n_2 + 1 \le n \le n_3$. Now, for any natural number n such that $n_2 \le n < n_3$, we obtain

$$|s_n - s| = s - s_n \le s - s_{n_2} \le b_{n_2}.$$

This procedure gives a choice $\delta_n \in \{-1, 1\}$ such that s_n converges to s, since $(b_n)_{n\in\mathbb{N}}$ converges to zero. □

From the above theorem we obtain the following result:

Corollary 2.7. *If the series $\sum_{n\in\mathbb{N}} a_n$ of real numbers is conditionally convergent and $s \in \overline{\mathbb{R}}$, then there exists a sequence of signs $(\delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \delta_n a_n = s$.*

3. The (S) property

Our main result asserts that the (R) property is equivalent to the (S) property. Before that we need that ideals with the (S) property are dense (Lemma 3.1) and not summable ideals (Lemma 3.2).

Lemma 3.1. *Let I be an admissible and proper ideal on $\mathbb{N}$. If I has the (S) property, then I is a dense ideal.*

Proof. Assume that I has the (S) property. Suppose that $A \subset \mathbb{N}$ with $A \notin I$; hence A is infinite. We write $A = \{n_k : k \in \mathbb{N}\}$ such that $n_1 < n_2 < \cdots < n_k < \cdots$. Consider the sequence $a = (a_n)_{n\in\mathbb{N}}$ defined in the following way:

$$a_{n_k} = \frac{(-1)^k}{k} \quad \text{if } k \in \mathbb{N}, \qquad a_n = 0 \quad \text{if } n \notin A.$$

As the series $\sum_{n\in\mathbb{N}} a_n$ is conditionally convergent and the ideal I has the (S) property, there exists a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} \delta_n a_n = \infty \quad \text{and} \quad N(\delta) = \{n \in \mathbb{N} : \delta_n = -1\} \in I.$$

Note that $N(\delta)$ is an infinite subset of $\mathbb{N}$ and

$$\sum_{n\in\mathbb{N}} \delta_n a_n = \sum_{n\in A} \delta_n a_n.$$

Hence $B = A \cap N(\delta) = \{n \in A : \delta_n = -1\}$ is infinite, $B \in I$ and $B \subset A$. Therefore I is a dense ideal. □

We use the following notation: given $A \subset \mathbb{N}$, denote by 1_A the characteristic function of A. For a sequence $a = (a_n)_{n\in\mathbb{N}}$ and a set $A \subset \mathbb{N}$, we write $1_A a = (1_A(n)a_n)_{n\in\mathbb{N}}$, which satisfies $1_A(n)a_n = a_n$ if $n \in A$, and $1_A(n)a_n = 0$ if $n \in \mathbb{N}\setminus A$. If $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} 1_A(n)a_n$ are convergent, we can write

$$\sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} 1_A(n)a_n + \sum_{n\in\mathbb{N}} 1_{\mathbb{N}\setminus A}(n)a_n.$$

Lemma 3.2. *Let I be an admissible and proper ideal on $\mathbb{N}$. If I is a summable ideal, then I does not have the (S) property.*

Proof. Assume that I is a summable ideal, that is $I = I_b = \{A \subset \mathbb{N} : \sum_{n\in A} b_n < \infty\}$, for a certain sequence $b = (b_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{n\in\mathbb{N}} b_n = \infty$. We consider two cases.

Case 1: the sequence $b = (b_n)_{n\in\mathbb{N}}$ is not convergent to 0. Then there exists $\varepsilon > 0$ such that $A = \{n \in \mathbb{N} : b_n \ge \varepsilon\}$ is an infinite subset of $\mathbb{N}$; hence $\sum_{n\in A} b_n = \infty$, so $A \notin I_b$. An infinite subset B of A also verifies $\sum_{n\in B} b_n = \infty$; therefore $B \notin I_b$. Consequently the ideal I_b is not dense. By Lemma 3.1 we have that the ideal I_b does not have the (S) property.

Case 2: the sequence $b = (b_n)_{n\in\mathbb{N}}$ is convergent to 0. By Theorem 2.6, there is a sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ such that the series $\sum_{n\in\mathbb{N}} \delta_n b_n$ is conditionally convergent. Let $\eta = (\eta_n)_{n\in\mathbb{N}}$ be any sequence of signs such that $\sum_{n\in\mathbb{N}} \eta_n \delta_n b_n = \infty$. If $N(\eta) \in I_b$, then $\sum_{n\in N(\eta)} b_n$ converges, so $\sum_{n\in N(\eta)} \delta_n b_n$ is absolutely convergent, and hence

$$\sum_{n\in N(\eta)} \delta_n b_n = -\sum_{n\in N(\eta)} \eta_n \delta_n b_n = -\sum_{n\in\mathbb{N}} \eta_n \delta_n 1_{N(\eta)}(n) b_n$$

is convergent; as $\sum_{n\in\mathbb{N}} \delta_n b_n$ is convergent, we obtain that

$$\sum_{n\in\mathbb{N}\setminus N(\eta)} \delta_n b_n = \sum_{n\in\mathbb{N}\setminus N(\eta)} \eta_n \delta_n b_n = \sum_{n\in\mathbb{N}} \eta_n \delta_n 1_{\mathbb{N}\setminus N(\eta)}(n) b_n$$

is also convergent and, therefore, the series

$$\sum_{n\in\mathbb{N}} \eta_n \delta_n b_n = \sum_{n\in\mathbb{N}} \eta_n \delta_n 1_{\mathbb{N}\setminus N(\eta)}(n) b_n - \sum_{n\in\mathbb{N}} \eta_n \delta_n 1_{N(\eta)}(n) b_n$$

is convergent, which yields a contradiction. Consequently $N(\eta) \notin I_b$. From this we obtain that I_b fails to have the (S) property with the series $\sum_{n\in\mathbb{N}} \delta_n b_n$ and $r = \infty$. □

Finally we give our main result.

Theorem 3.3. *Let I be an admissible and proper ideal on $\mathbb{N}$. Then*

$$I \text{ has the } (R) \text{ property } \iff I \text{ has the } (S) \text{ property }.$$

Proof. ($\Longrightarrow$) Assume that I has the (R) property, but I does not have the (S) property. There exist a conditionally convergent series $\sum_{n\in\mathbb{N}} a_n$ and $r \in \overline{\mathbb{R}}$ such that for every sequence of signs $\delta = (\delta_n)_{n\in\mathbb{N}}$ verifying $\sum_{n\in\mathbb{N}} \delta_n a_n = r$, we have that $N(\delta) \notin I$. Moreover, by Theorem 2.4, as I has the (W) property, then there exists $A \in I$ such that the series $\sum_{n\in A} a_n$ is also conditionally convergent. Since $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} 1_A(n) a_n$ are convergent, we can write

$$\sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} 1_A(n) a_n + \sum_{n\in\mathbb{N}} 1_{\mathbb{N}\setminus A}(n) a_n.$$

Moreover, there exists a sequence of signs $\eta = (\eta_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} \eta_n 1_A(n) a_n = r - \sum_{n\in\mathbb{N}} 1_{\mathbb{N}\setminus A}(n) a_n.$$

Note that we can choose $\eta_n = 1$ if $n \notin A$; hence $N(\eta) \subset A \in I$, so $N(\eta) \in I$. Now we write

$$\sum_{n\in\mathbb{N}} \eta_n a_n = \sum_{n\in\mathbb{N}} \eta_n 1_A(n) a_n + \sum_{n\in\mathbb{N}} 1_{\mathbb{N}\setminus A}(n) a_n = r,$$

and we obtain a contradiction.

($\Longleftarrow$) Assume that I does not have the (R) property, and hence I can be extended to a summable ideal $K \supset I$, by Theorem 2.4. Then K does not have the (S) property, by Lemma 3.2; hence I does not have the (S) property. □

Corollary 3.4. *The ideal $\mathcal{I}_d$ of all the sets of asymptotic density zero has the (S) property.*

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